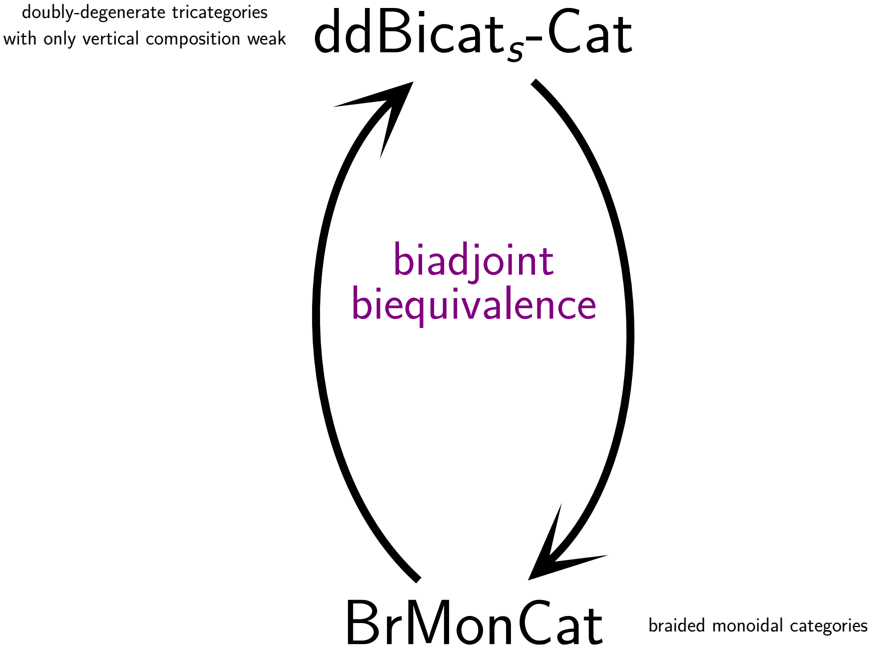
Weak vertical composition

Eugenia Cheng, School of the Art Institute of Chicago & Alexander S. Corner, Sheffield Hallam University


doubly-degenerate tricategories
with only vertical composition weak
ddBicat$_s$-Cat
biadjoint
biequivalence
BrMonCat
braided monoidal categories

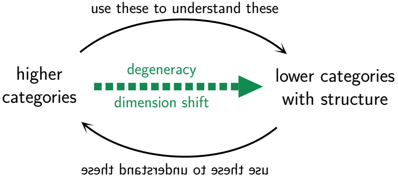
Background ideas

Degeneracy, Totalities, Icons, Semi-Strictness, Eckmann-Hilton, Braidings

Degeneracy

[BD]

Idea: in k-degenerate n-categories the bottom k dimensions are trivial

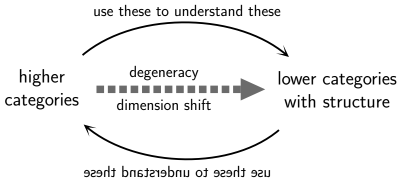


1-cats	1-degen →	monoids
2-cats	1-degen →	monoidal cats
2-cats	2-degen →	commutative monoids
3-cats	2-degen →	braided monoidal cats

Degeneracy

[BD]

Idea: in k-degenerate n-categories
the bottom k dimensions are trivial



1-cats	—1-degen→	monoids
2-cats	—1-degen→	monoidal cats
2-cats	—2-degen→	commutative monoids
3-cats	—2-degen→	braided monoidal cats

Degeneracy and totalities

Doubly-degenerate tricategories "are"
braided monoidal categories
but the totalities are more difficult. [CG]

4-category of 3-categories ⟵ ? ⟶ 2-category of braided monoidal categories

- LHS has the wrong dimensions for comparison.
- LHS maps are too weak for degenerate versions: distinguished invertible elements arise.
- We use iterated icons to fix both issues.

iconic 2-category of 3-categories ⟷ 2-category of braided monoidal categories

We can now take the full sub-2-category of doubly-degenerate 3-categories.

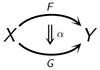
Icons

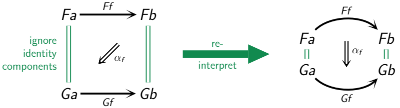
"Identity Component Oplax Natural" [Lack]

Idea: make a convenient 2-category of bicategories

- Bicategories naturally form a tricategory that doesn't truncate to a 2-category.
- We make a 2-category by changing the 2-cells.

An icon between morphisms of bicategories exists only when F and G agree on 0-cells.





- Components are 2-cells only, so compose strictly.

Degeneracy and totalities

Doubly-degenerate tricategories "are" braided monoidal categories but the totalities are more difficult. [CG]

4-category of 3-categories $\overset{?}{\longleftrightarrow}$ 2-category of braided monoidal categories

- LHS has the wrong dimensions for comparison.
- LHS maps are too weak for degenerate versions: distinguished invertible elements arise.
- We use iterated icons to fix both issues.

iconic 2-category of 3-categories $\longleftrightarrow$ 2-category of braided monoidal categories

We can now take the full sub-2-category of doubly-degenerate 3-categories.

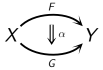
Icons

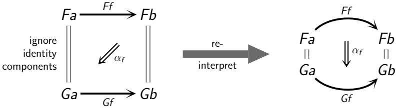
"Identity Component Oplax Natural" [Lack]

Idea: make a convenient 2-category of bicategories

- Bicategories naturally form a tricategory that doesn't truncate to a 2-category.
- We make a 2-category by changing the 2-cells.

An icon between morphisms of bicategories exists only when F and G agree on 0-cells.

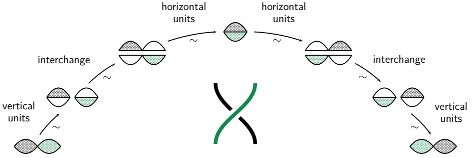




- Components are 2-cells only, so compose strictly.

Eckmann–Hilton and braiding

In a doubly-degenerate tricategory we get a braiding from a weak Eckmann–Hilton argument.



- This depends on having only one 0-cell and 1-cell.
- For doubly-degenerate bicategories this is strict, producing commutative monoids.
- If 1-cell units are weak the argument is more complicated. [JS]

Flavours of semi-strictness

Weak interchange

[GPS]

- Every tricategory is equivalent to a Gray-category.
- Idea: everything is strict except interchange.

Weak horizontal units

[JK]

- Given any braided monoidal category B there is a monoidal 2-category X with weak unit I such that $X(I, I)$ is braided monoidal equivalent to B.

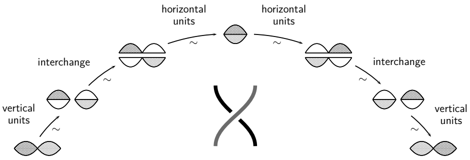
Weak vertical composition

[CC]

- Any braided monoidal category B arises from a doubly-degenerate tricategory with everything strict except vertical composition.

Eckmann–Hilton and braiding

In a doubly-degenerate tricategory we get a braiding from a weak Eckmann–Hilton argument.



- This depends on having only one 0-cell and 1-cell.
- For doubly-degenerate bicategories this is strict, producing commutative monoids.
- If 1-cell units are weak the argument is more complicated. [JS]

Flavours of semi-strictness

Weak interchange

[GPS]

- Every tricategory is equivalent to a Gray-category.
- Idea: everything is strict except interchange.

Weak horizontal units

[JK]

- Given any braided monoidal category B there is a monoidal 2-category X with weak unit I such that $X(I, I)$ is braided monoidal equivalent to B.

Weak vertical composition

[CC]

- Any braided monoidal category B arises from a doubly-degenerate tricategory with everything strict except vertical composition.

Bicat_s-categories

Idea: tricategories with
only vertical composition weak

- Bicat_s is the category of bicategories and strict functors, with cartesian monoidal structure.

Then a category enriched in Bicat_s has:

- strict composition functors → strict interchange
- strict enrichment → strict horizontal composition
- weak composition in hom bicategories → weak vertical composition

---

We make an iconic 2-category totality of these.

Comparing totalities

Iconic totalities, Distributive laws, Weak maps

Iconic totality construction

Idea: make an iconic 2-category of Bicat_s-categories as strict algebras for a 2-monad on Cat-Gph-Gph

---

Cat-Gph-Gph is a 2-category with

- 0-cells: 3-globular sets where the 2- and 3-cells form a category
- 1-cells: morphisms of such
- 2-cells: "ico-iconic", where the source and target morphisms must agree on 0- and 1-cells

---

We define strict 2-monads on Cat-Gph-Gph:

- H for (strict) horizontal composition
- V for (weak) vertical composition

and a 2-distributive law $VH \Longrightarrow HV$ with

$$HV\text{-Alg} \;\cong\; \mathrm{Bicat}_s\text{-}\mathrm{Cat}_s$$

isomorphism of 2-categories

- This deals with strict maps only.
- This automatically constructs ico-iconic 2-cells.
- We provide greater generality via operad actions, with a view to future work.

Iconic totality construction

Idea: make an iconic 2-category of Bicat_s-categories as strict algebras for a 2-monad on Cat-Gph-Gph

Cat-Gph-Gph is a 2-category with

- 0-cells: 3-globular sets where the 2- and 3-cells form a category
- 1-cells: morphisms of such
- 2-cells: "ico-iconic", where the source and target morphisms must agree on 0- and 1-cells

We define strict 2-monads on Cat-Gph-Gph:

- H for (strict) horizontal composition
- V for (weak) vertical composition

and a 2-distributive law $VH \Longrightarrow HV$ with

$$HV\text{-Alg} \cong \mathrm{Bicat}_s\text{-}\mathrm{Cat}_s$$

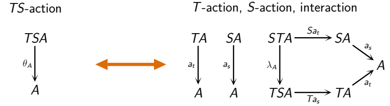
isomorphism of 2-categories

- This deals with strict maps only.
- This automatically constructs ico-iconic 2-cells.
- We provide greater generality via operad actions, with a view to future work.

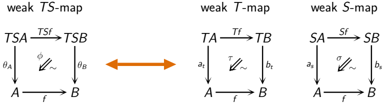
Weak maps via distributive laws

Idea: given a distributive law of 2-monads $ST \overset{\lambda}{\Longrightarrow} TS$ we study the 2-category $TS\text{-Alg}_w$ via S and T

Strict algebras



Weak maps



interaction axiom not pictured

Transformations

no interaction axiom

Weak maps unravelled

We construct the desired 2-category totality

$\mathrm{Bicat}_s\text{-}\mathrm{Cat} := HV\text{-}\mathrm{Alg}_w$

- We restrict to doubly-degenerate Bicat_s-categories.
- Maps have iconic constraints so are appropriately semi-strict for the doubly-degenerate case.

Doubly-degenerate HV-algebras

H-action → strict horizontal tensor $a|b$

V-action → weak vertical tensor $\frac{a}{b}$

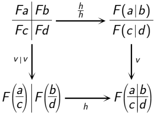
interaction → strict interchange

Weak maps of doubly-degenerate HV-algebras

weak H-map → $Fa|Fb \xrightarrow{h} F(a|b)$

weak V-map → $\frac{Fa}{Fb} \xrightarrow{v} F\left(\frac{a}{b}\right)$

interaction →



- v can always be used to reconstruct h
- V-transformation implies H-transformation

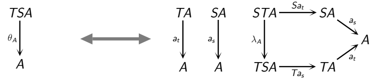
Weak maps via distributive laws

Idea: given a distributive law of 2-monads $ST \overset{\lambda}{\Longrightarrow} TS$ we study the 2-category $TS\text{-}\mathrm{Alg}_w$ via S and T

Strict algebras

TS-action

T-action, S-action, interaction

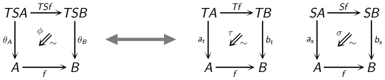


Weak maps

weak TS-map

weak T-map

weak S-map



interaction axiom not pictured

Transformations

TS-transformation

T- and S-transformation

no interaction axiom

Weak maps unravelled

We construct the desired 2-category totality

$$\mathrm{Bicat}_s\text{-}\mathrm{Cat} := HV\text{-}\mathrm{Alg}_w$$

- We restrict to doubly-degenerate Bicat_s-categories.
- Maps have iconic constraints so are appropriately semi-strict for the doubly-degenerate case.

Doubly-degenerate HV-algebras

H-action	⟶	strict horizontal tensor $a \mid b$
V-action	⟶	weak vertical tensor $\frac{a}{b}$
interaction	⟶	strict interchange

Weak maps of doubly-degenerate HV-algebras

weak H-map ⟶ $Fa \mid Fb \xrightarrow{h} F(a \mid b)$

weak V-map ⟶ $\dfrac{Fa}{Fb} \xrightarrow{v} F\left(\dfrac{a}{b}\right)$

interaction ⟶

$$\begin{array}{ccc} \begin{array}{c|c} Fa & Fb \\ \hline Fc & Fd \end{array} & \xrightarrow{\frac{h}{h}} & \begin{array}{c} F(a \mid b) \\ \hline F(c \mid d) \end{array} \\ \downarrow{\scriptstyle v \mid v} & & \downarrow{\scriptstyle v} \\ F\left(\frac{a}{c}\right) \Big| F\left(\frac{b}{d}\right) & \xrightarrow[h]{} & F\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \end{array}$$

- v can always be used to reconstruct h
- V-transformation implies H-transformation

Comparison with braided monoidal categories

We construct a 2-functor

$$\mathrm{ddBicat}_s\text{-}\mathrm{Cat} \xrightarrow{U} \mathrm{BrMonCat}$$

- On 0-cells: use vertical tensor product (V-action), construct braiding by weak Eckmann–Hilton.
- On 1-cells: use weak V-map structure, interaction axiom gives braid axiom.
- On 2-cells: use V-transformation structure.

Biessentially surjective on 0-cells [CC]

- Previous work.

Locally surjective on 1-cells

- Reconstruct h from v.

$$\begin{array}{ccc} Fa \mid Fb & \xrightarrow{h} & F(a \mid b) \\ {\scriptstyle \alpha}\downarrow\wr & & \wr\downarrow{\scriptstyle F\alpha} \\ \dfrac{Fa}{Fb} & \xrightarrow[v]{} & F\left(\dfrac{a}{b}\right) \end{array}$$

half an Eckmann–Hilton argument

- Interaction axiom follows from braid axiom.

Locally full and faithful on 2-cells

- A transformation is vertically monoidal if and only if it is vertically and horizontally monoidal.

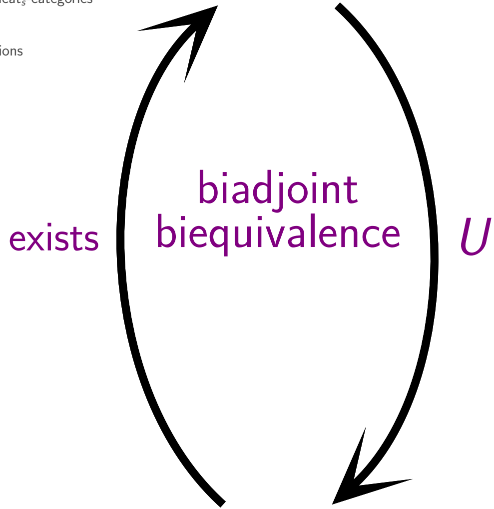
doubly-degenerate tricategories
with only vertical composition weak

ddBicat$_s$-Cat

0-cells: doubly-degenerate Bicat$_s$-categories

1-cells: weak maps

2-cells: icon-like transformations



Corollary

Biadjoint biequivalence

follows from U being a pointwise biequivalence and [Gurski]

Theorem

U is a pointwise biequivalence

- biessentially surjective on 0-cells
- locally [essentially] surjective on 1-cells
- locally full and faithful on 2-cells

BrMonCat

0-cells: braided weak monoidal categories

1-cells: braided weak monoidal functors

2-cells: monoidal transformations

Future and related work

- An analogous analysis for doubly-degenerate Trimble tricategories. These are weakened by operad actions; in the present work we express bicategories via operad actions to lay some groundwork for the generalisation.
- Generalisation to $(n-1)$-degenerate n-categories; these should all be categories with extra structure, with 2-category totalities.
- Constructing a pseudo-inverse for U, and deducing a free 2-functor from categories via the free braided monoidal category 2-functor.
- Relation to Cheng–Garner constructing operads for k-degenerate $(n+k)$-categories.

References

[BD] J. Baez, J. Dolan. Higher-dimensional algebra and topological quantum field theory. *J. Math. Phys.*, 36:6073–6105, 1995.

[CC] E. Cheng, A. S. Corner. Weak vertical composition (2022). E-print 2212.11865. (To appear: *Theory Appl. Categories.*)

[CC2] E. Cheng, A. S. Corner. Weak vertical composition II: totalities (2023). E-print 2308.09806.

[CG] E. Cheng, N. Gurski. Iterated icons. *Theory Appl. Categories*, 29:929–977, 2014.

[GPS] R. Gordon, A. J. Power, R. Street. *Coherence for tricategories*. Mem. Amer. Math. Soc. 117 (1995) no. 558.

[Gurski] N. Gurski. Biequivalences in tricategories. *Theory Appl. Categories*, 26(14):349–384, 2012.

[JK] A. Joyal, J. Kock. Weak units and homotopy 3-types. In *Categories in Algebra, Geometry and Mathematical Physics*, Contemporary Mathematics (2007) no. 431

[JK] A. Joyal, R. Street. Braided tensor categories. *Adv. Math.*, 102:20–78, 1983.

[Lack] S. Lack. Icons. *Appl. Categ. Struct.*, 18(3):289–307, 2010.