A Higher-Dimensional Eckmann–Hilton Argument

Eugenia Cheng, Alex Corner

School of the Art Institute Chicago, Sheffield Hallam University

Slides: alex-corner.github.io/slides/corner-ct24.pdf

Plan

Aim: Describe the 3-fold generalisation of the Eckmann–Hilton argument for 3-degenerate 4-categories...

Slides: alex-corner.github.io/slides/corner-ct24.pdf

Plan

Aim: Describe the 3-fold generalisation of the Eckmann–Hilton argument for 3-degenerate 4-categories...

...with a mind to develop this in generality for $(n-1)$-degenerate n-categories.

Slides: alex-corner.github.io/slides/corner-ct24.pdf

Plan

Aim: Describe the 3-fold generalisation of the Eckmann–Hilton argument for 3-degenerate 4-categories...

...with a mind to develop this in generality for $(n-1)$-degenerate n-categories.

Section n: degenerate n-categories $(1 \leq n \leq 4)$

Slides: alex-corner.github.io/slides/corner-ct24.pdf

The Concept of Degeneracy

The Concept of Degeneracy

The Concept of Degeneracy

category	monoid
0-cells	
1-cells	
composition	
identity	

The Concept of Degeneracy

category		monoid
0-cells} trivial		-
1-cells	→	elements
composition	→	multiplication
identity	→	unit

The Concept of Degeneracy

category		monoid
0-cells} trivial		-
1-cells	⟶	elements
composition	⟶	multiplication
identity	⟶	unit

Idea: In a k-degenerate n-category, the bottom k dimensions are trivial.

n-category	k-d-n-category
0-cells	
1-cells	
$\vdots$	
$(k-1)$-cells	
k-cells	
$\vdots$	
$(n-1)$-cells	
n-cells	

The Concept of Degeneracy

category		monoid
0-cells} trivial		-
1-cells	⟶	elements
composition	⟶	multiplication
identity	⟶	unit

Idea: In a k-degenerate n-category, the bottom k dimensions are trivial.

n-category		k-d-n-category
0-cells	trivial	-
1-cells	trivial	-
⋮	trivial	⋮
$(k-1)$-cells	trivial	-
k-cells		
⋮		
$(n-1)$-cells		
n-cells		

The Concept of Degeneracy

category		monoid
0-cells } trivial		-
1-cells	⟶	elements
composition	⟶	multiplication
identity	⟶	unit

Idea: In a k-degenerate n-category, the bottom k dimensions are trivial.

n-category		k-d-n-category
0-cells	} trivial	-
1-cells		-
$\vdots$		$\vdots$
$(k-1)$-cells		-
k-cells	⟶	0-cells
$\vdots$	$\vdots$	$\vdots$
$(n-1)$-cells	⟶	$(n-k-1)$-cells
n-cells	⟶	$(n-k)$-cells

The Concept of Degeneracy

category		monoid
0-cells } trivial		-
1-cells	⟶	elements
composition	⟶	multiplication
identity	⟶	unit

Idea: In a k-degenerate n-category, the bottom k dimensions are trivial.

n-category		k-d-n-category
0-cells	} trivial	-
1-cells		-
⋮		⋮
$(k-1)$-cells		-
k-cells	⟶	0-cells
⋮	⋮	⋮
$(n-1)$-cells	⟶	$(n-k-1)$-cells
n-cells	⟶	$(n-k)$-cells

Idea: A k-degenerate n-category 'is' an $(n-k)$-category with extra structure.

The Concept of Degeneracy

category		monoid
0-cells } trivial		-
1-cells	⟶	elements
composition	⟶	multiplication
identity	⟶	unit

Idea: In a k-degenerate n-category, the bottom k dimensions are trivial.

n-category		k-d-n-category
0-cells	trivial	-
1-cells	trivial	-
⋮	trivial	⋮
$(k-1)$-cells	trivial	-
k-cells	⟶	0-cells
⋮	⋮	⋮
$(n-1)$-cells	⟶	$(n-k-1)$-cells
n-cells	⟶	$(n-k)$-cells

Idea: A k-degenerate n-category ‘is’ an $(n-k)$-category with extra structure.

A Question of Totalities

The Concept of Degeneracy

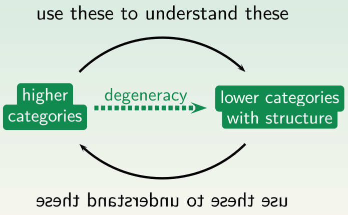
category		monoid
0-cells } trivial		-
1-cells	⟶	elements
composition	⟶	multiplication
identity	⟶	unit

Idea: In a k-degenerate n-category, the bottom k dimensions are trivial.

n-category		k-d-n-category
0-cells	trivial	-
1-cells	trivial	-
⋮	trivial	⋮
$(k-1)$-cells	trivial	-
k-cells	⟶	0-cells
⋮	⋮	⋮
$(n-1)$-cells	⟶	$(n-k-1)$-cells
n-cells	⟶	$(n-k)$-cells

Idea: A k-degenerate n-category 'is' an $(n-k)$-category with extra structure.

A Question of Totalities



The Concept of Degeneracy

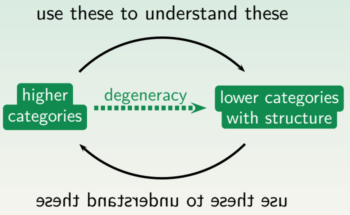
category		monoid
0-cells } trivial		-
1-cells	$\longrightarrow$	elements
composition	$\longrightarrow$	multiplication
identity	$\longrightarrow$	unit

Idea: In a k-degenerate n-category, the bottom k dimensions are trivial.

n-category		k-d-n-category
0-cells	trivial	-
1-cells	trivial	-
$\vdots$	trivial	$\vdots$
$(k-1)$-cells	trivial	-
k-cells	$\longrightarrow$	0-cells
$\vdots$	$\vdots$	$\vdots$
$(n-1)$-cells	$\longrightarrow$	$(n-k-1)$-cells
n-cells	$\longrightarrow$	$(n-k)$-cells

Idea: A k-degenerate n-category 'is' an $(n-k)$-category with extra structure.

A Question of Totalities



Objects	Totality
monoids	category
categories	2-category
monoidal categories	2-category
2-categories	3-category

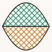
Idea: **d-Cat** is a full sub-2-category of **Cat**. We can 'truncate' **d-Cat** to a category: discard the natural transformations.

Problem: **d-2-Cat** is a full sub-3-category of **2-Cat**. We can't 'truncate' **d-2-Cat** to a 2-category: this is fixed using icons.

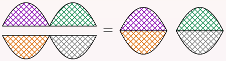
1-degenerate 2-categories

2-category		monoidal category
0-cells} trivial		-
1-cells	⟶	objects
2-cells	⟶	morphisms
hor. composites	⟶	$\otimes$

vert. composites	⟶	composition

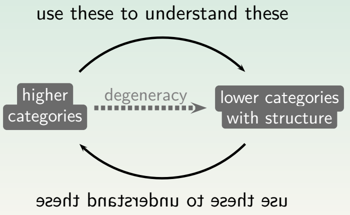


identity	⟶	unit object
interchange	⟶	interchange



$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

A Question of Totalities



Objects	Totality
monoids	category
categories	2-category
monoidal categories	2-category
2-categories	3-category

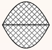
Idea: **d-Cat** is a full sub-2-category of **Cat**.
We can ‘truncate’ **d-Cat** to a category:
discard the natural transformations.

Problem: **d-2-Cat** is a full sub-3-category of **2-Cat**.
We can’t ‘truncate’ **d-2-Cat** to a 2-category:
this is fixed using icons.

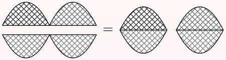
1-degenerate 2-categories

2-category		monoidal category
0-cells } trivial		-
1-cells	⟶	objects
2-cells	⟶	morphisms
hor. composites	⟶	$\otimes$

vert. composites	⟶	composition



identity	⟶	unit object
interchange	⟶	interchange



$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

2-degenerate 2-categories

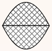
2-category		comm. monoid
0-cells } trivial		-
1-cells } trivial		-
2-cells	⟶	elements
hor. composites	⟶	multiplication $*$
vert. composites	⟶	multiplication $\circ$
identity	⟶	unit for $*$ and $\circ$
interchange	⟶	interchange

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

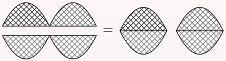
1-degenerate 2-categories

2-category		monoidal category
0-cells } trivial		-
1-cells	⟶	objects
2-cells	⟶	morphisms
hor. composites	⟶	$\otimes$

vert. composites	⟶	composition



identity	⟶	unit object
interchange	⟶	interchange



$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

2-degenerate 2-categories

2-category		comm. monoid
0-cells } trivial		-
1-cells } trivial		-
2-cells	⟶	elements
hor. composites	⟶	multiplication $*$
vert. composites	⟶	multiplication $\circ$
identity	⟶	unit for $*$ and $\circ$
interchange	⟶	interchange

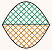
$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

Idea: $\circ$ and $*$ are the same and commutative.
This is the Eckmann–Hilton argument:

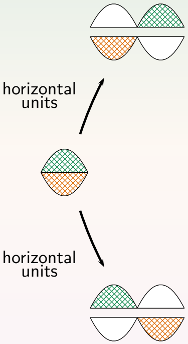
$$\begin{aligned} a \circ b &= (1 * a) \circ (b * 1) \\ &= (1 \circ b) * (a \circ 1) \\ &= b * a \\ &= (b \circ 1) * (1 \circ a) \\ &= (b * 1) \circ (1 * a) \\ &= b \circ a \end{aligned}$$

Eckmann–Hilton Argument

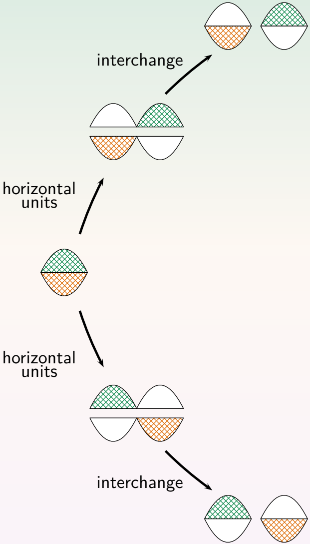
Eckmann–Hilton Argument



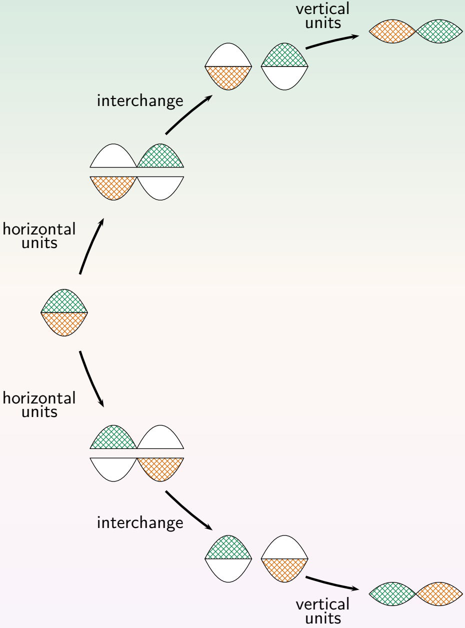
Eckmann–Hilton Argument



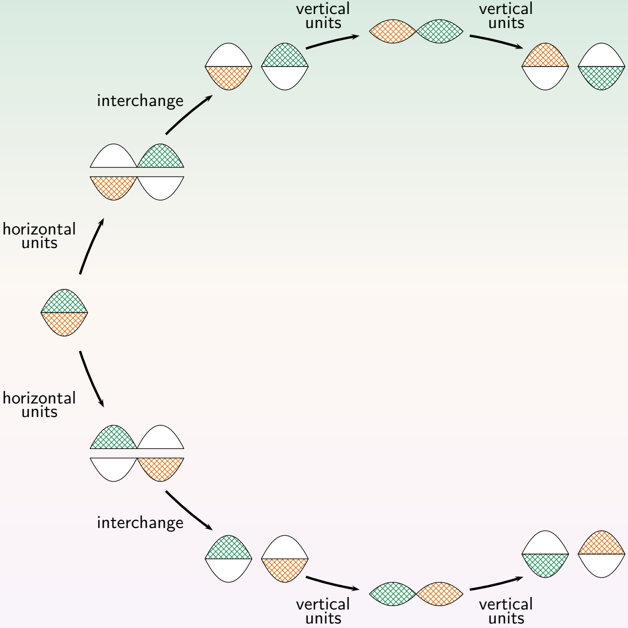
Eckmann–Hilton Argument



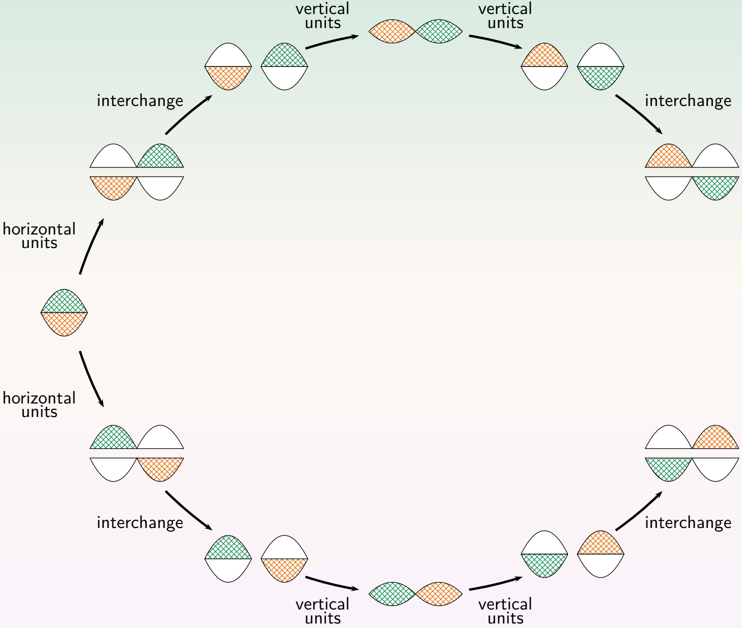
Eckmann–Hilton Argument



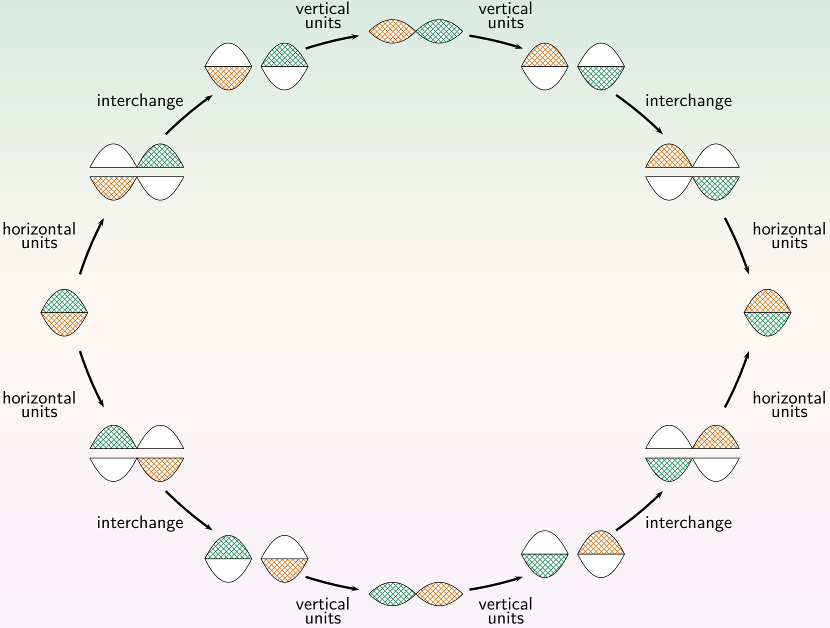
Eckmann–Hilton Argument



Eckmann–Hilton Argument



Eckmann–Hilton Argument



The Periodic Table of n-Categories

set

The Periodic Table of n-Categories

set

category

monoid

The Periodic Table of n-Categories

The Periodic Table of n-Categories

set	category	2-category	3-category
monoid	monoidal category		
commutative monoid	braided monoidal category		

This is what we were interested in: 2-degenerate 3-categories.

The Periodic Table of n-Categories

set

category

2-category

3-category

4-category

monoid

monoidal category

commutative monoid

braided monoidal category

This is what we were interested in: 2-degenerate 3-categories.

symmetric monoidal category

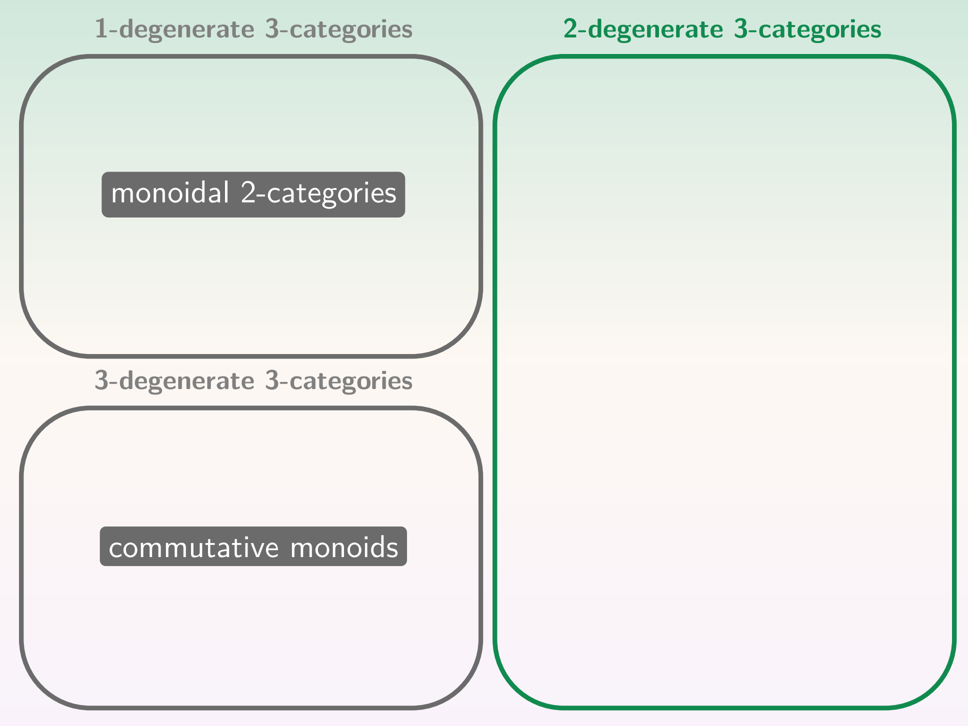
We want to repeat the story for 3-degenerate 4-categories.

1-degenerate 3-categories

monoidal 2-categories

3-degenerate 3-categories

commutative monoids


1-degenerate 3-categories
monoidal 2-categories
3-degenerate 3-categories
commutative monoids
2-degenerate 3-categories

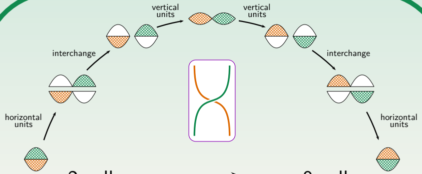
1-degenerate 3-categories

monoidal 2-categories

3-degenerate 3-categories

commutative monoids

2-degenerate 3-categories



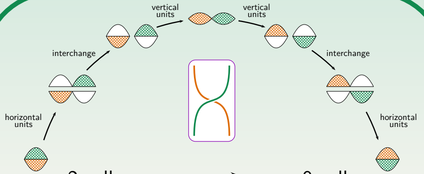
2-cells	⟶	0-cells
3-cells	⟶	1-cells
composites	⟶	$\otimes$s
interchange	⟶	braiding

braided monoidal categories

1-degenerate 3-categories

monoidal 2-categories

2-degenerate 3-categories

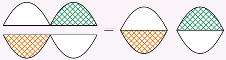


2-cells	⟶	0-cells
3-cells	⟶	1-cells
composites	⟶	$\otimes$s
int E–H ge	⟶	braiding

braided monoidal categories

Coherence in 3 flavours

	vertical composition	horizontal units	interchange
GPS	strict	strict	weak
JK	strict	weak	strict
CC	weak	strict	strict



3-degenerate 3-categories

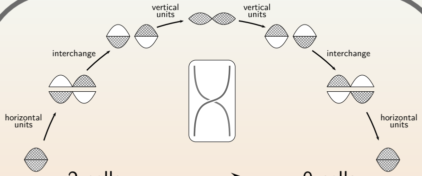
commutative monoids

3-degenerate 4-categories

symmetric monoidal categories

semi-strict 4-categories

2-degenerate 3-categories



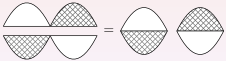
2-cells	$\longrightarrow$	0-cells
3-cells	$\longrightarrow$	1-cells
composites	$\longrightarrow$	$\otimes$s
interchange	$\longrightarrow$	braiding

E–H

braided monoidal categories

Coherence in 3 flavours

	vertical composition	horizontal units	interchange
GPS	strict	strict	weak
JK	strict	weak	strict
CC	weak	strict	strict



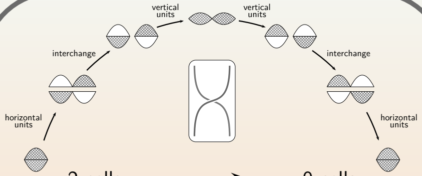
3-degenerate 4-categories

symmetric monoidal categories

semi-strict 4-categories

composition of 1-cells
composition of 2-cells } strict
interchange

2-degenerate 3-categories

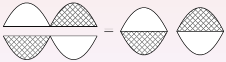


2-cells	$\longrightarrow$	0-cells
3-cells	$\longrightarrow$	1-cells
composites	$\longrightarrow$	$\otimes$s
interchange (E–H)	$\longrightarrow$	braiding

braided monoidal categories

Coherence in 3 flavours

	vertical composition	horizontal units	interchange
GPS	strict	strict	weak
JK	strict	weak	strict
CC	weak	strict	strict



3-degenerate 4-categories

symmetric monoidal categories

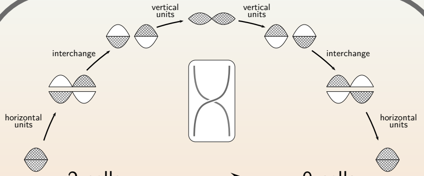
semi-strict 4-categories

composition of 1-cells	strict
composition of 2-cells	strict
interchange	strict

composition of 3-cells

0-composition	weak
1-composition	strict
2-composition	weak

2-degenerate 3-categories

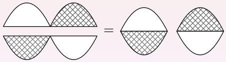


2-cells	→	0-cells
3-cells	→	1-cells
composites	→	⊗s
interchange E–H	→	braiding

braided monoidal categories

Coherence in 3 flavours

	vertical composition	horizontal units	interchange
GPS	strict	strict	weak
JK	strict	weak	strict
CC	weak	strict	strict



3-degenerate 4-categories

symmetric monoidal categories

semi-strict 4-categories

composition of 1-cells
composition of 2-cells
interchange
} strict

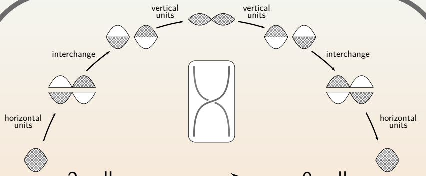
composition of 3-cells

0-composition	weak
1-composition	strict
2-composition	weak

Mix of strict and weak enrichment:

- Vertically weak tricategories: $\mathbf{Bicat}_s$-**Cat**.

2-degenerate 3-categories

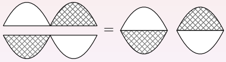


2-cells	⟶	0-cells
3-cells	⟶	1-cells
composites	⟶	⊗s
int E–H ge	⟶	braiding

braided monoidal categories

Coherence in 3 flavours

	vertical composition	horizontal units	interchange
GPS	strict	strict	weak
JK	strict	weak	strict
CC	weak	strict	strict



3-degenerate 4-categories

symmetric monoidal categories

semi-strict 4-categories

composition of 1-cells
composition of 2-cells
interchange
} strict

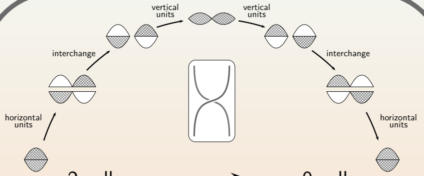
composition of 3-cells

0-composition	weak
1-composition	strict
2-composition	weak

Mix of strict and weak enrichment:

- Vertically weak tricategories: $\mathbf{Bicat}_s$-**Cat**.
- Semi-strict 4-categories: $\mathbf{Bicat}_s$-**Cat**-**wCat**.
    - Use iterated icons, 2-monads, iterated distributive laws on **Cat**-**Gph**-**Gph**.

2-degenerate 3-categories

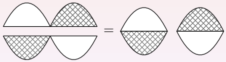


2-cells	⟶	0-cells
3-cells	⟶	1-cells
composites	⟶	$\otimes$s
int E–H ge	⟶	braiding

braided monoidal categories

Coherence in 3 flavours

	vertical composition	horizontal units	interchange
GPS	strict	strict	weak
JK	strict	weak	strict
CC	weak	strict	strict



3-degenerate 4-categories

symmetric monoidal categories

semi-strict 4-categories

composition of 1-cells
composition of 2-cells
interchange
} strict

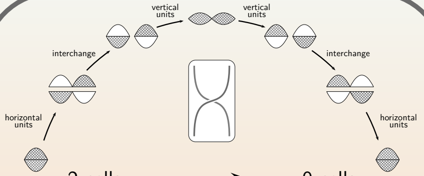
composition of 3-cells

0-composition	weak
1-composition	strict
2-composition	weak

Mix of strict and weak enrichment:

- Vertically weak tricategories: $\mathbf{Bicat}_s$-**Cat**.
- Semi-strict 4-categories: $\mathbf{Bicat}_s$-**Cat**-**wCat**.
  - Use iterated icons, 2-monads, iterated distributive laws on **Cat**-**Gph**-**Gph**.
- Triply-degenerate: we can produce a symmetric monoidal category.
- Future: Obtain full coherence result and comparison of totalities like before.

2-degenerate 3-categories

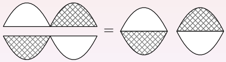


2-cells	⟶	0-cells
3-cells	⟶	1-cells
composites	⟶	⊗s
int E–H ge	⟶	braiding

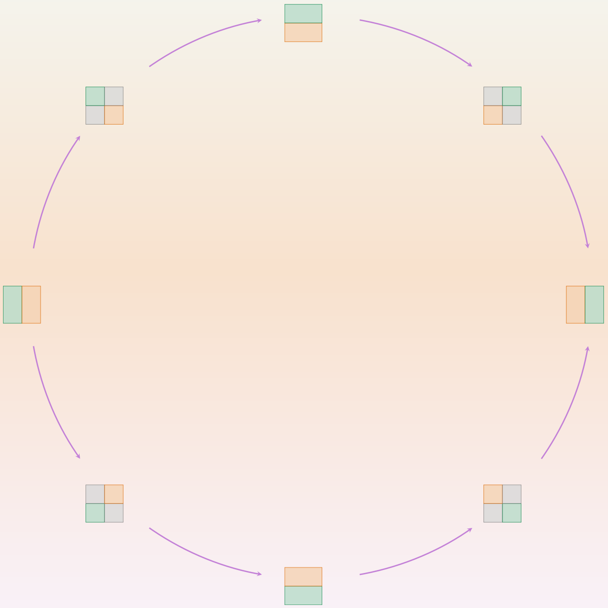
braided monoidal categories

Coherence in 3 flavours

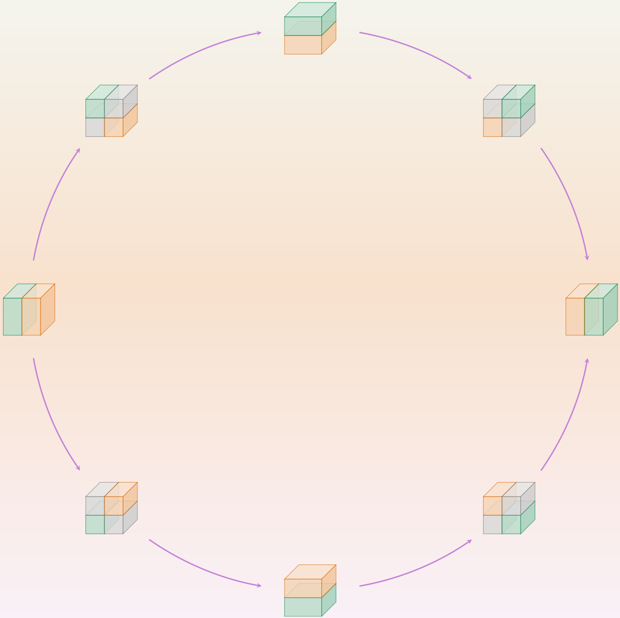
	vertical composition	**horizontal units**	**interchange**
GPS	strict	strict	weak
JK	strict	weak	strict
CC	weak	strict	strict



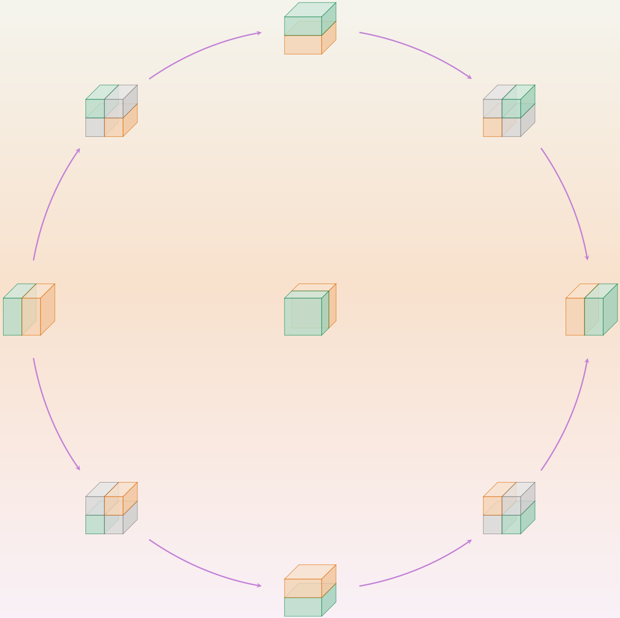
The Eckmann–Hilton Clock



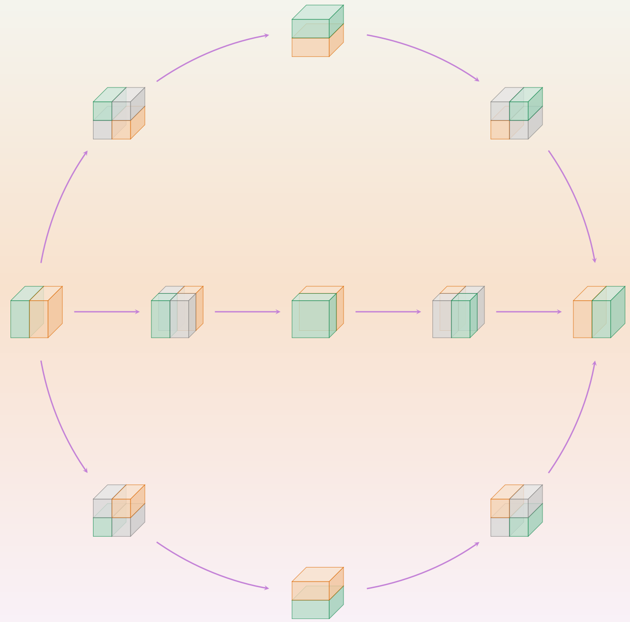
The Eckmann–Hilton Sphere



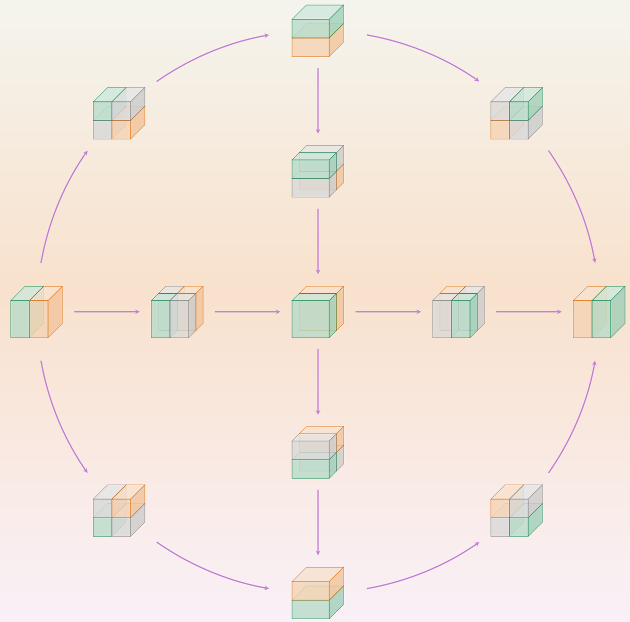
The Eckmann–Hilton Sphere



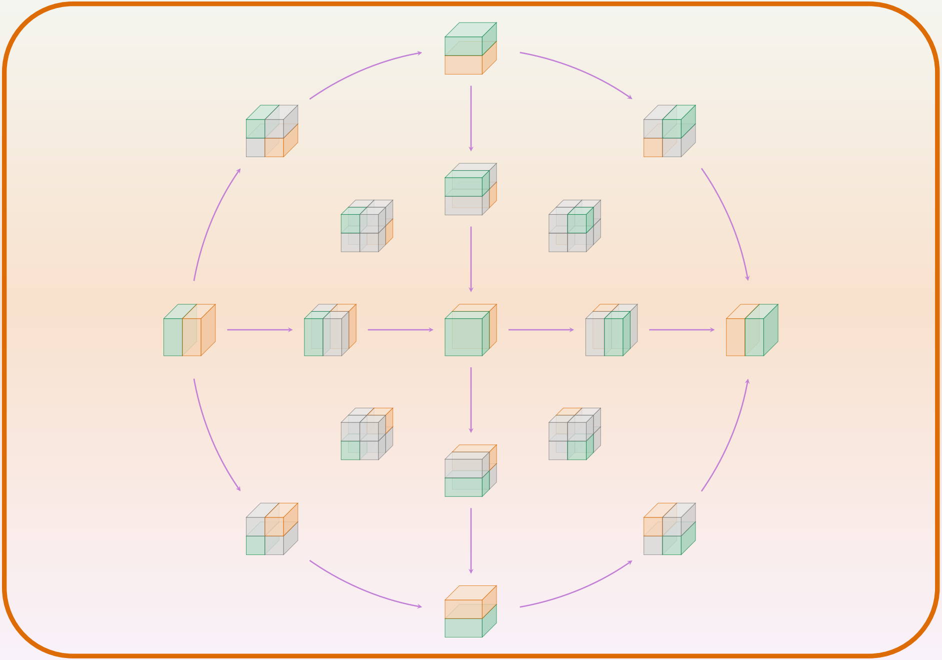
The Eckmann–Hilton Sphere



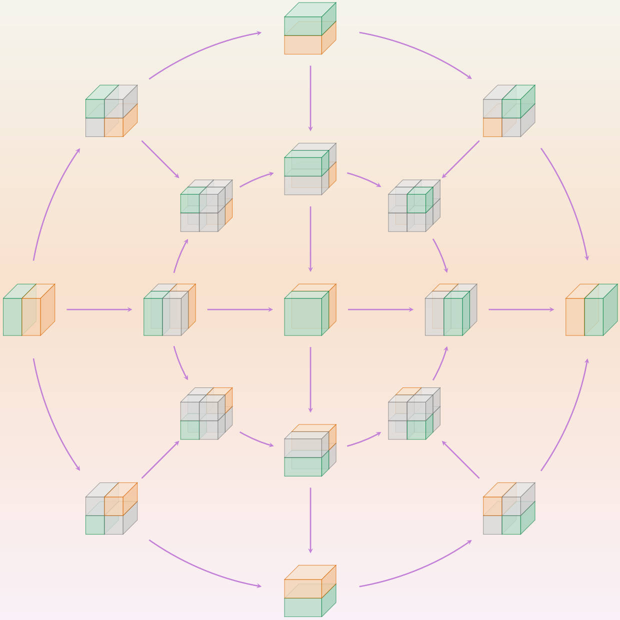
The Eckmann–Hilton Sphere



The Eckmann–Hilton Sphere

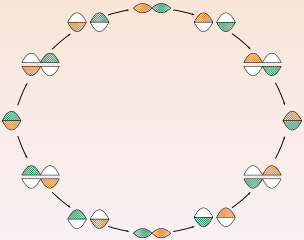


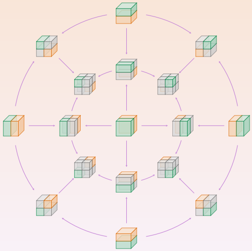
The Eckmann–Hilton Sphere



The Eckmann–Hilton Arguments

The Eckmann–Hilton argument in the context of degenerate higher categories appears as a heirarchy of results:

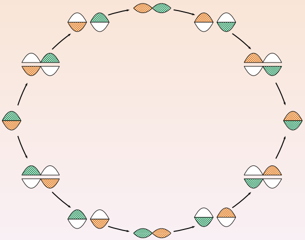


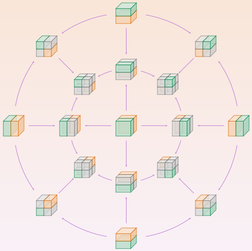


The Eckmann–Hilton Arguments

The Eckmann–Hilton argument in the context of degenerate higher categories appears as a heirarchy of results:

- Let $\mathcal{A}$ be a set with two monoid structures satisfying interchange, then the Eckmann–Hilton argument shows that the two monoid structures are the same and commutative.

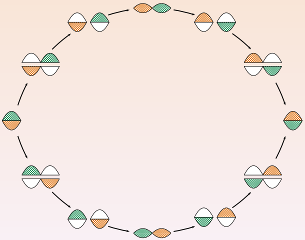


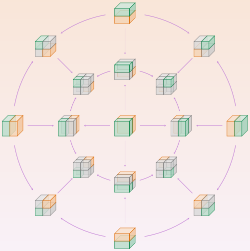


The Eckmann–Hilton Arguments

The Eckmann–Hilton argument in the context of degenerate higher categories appears as a heirarchy of results:

- Let $\mathcal{A}$ be a set with two monoid structures satisfying interchange, then the Eckmann–Hilton argument shows that the two monoid structures are the same and commutative.
- Let $\mathcal{A}$ be a category with two monoidal structures satisfying interchange, then the *weak* Eckmann–Hilton argument produces a braiding such that the monoidal products are weakly commutative and isomorphic.

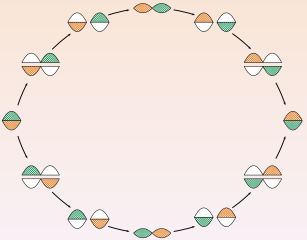


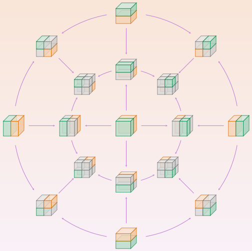


The Eckmann–Hilton Arguments

The Eckmann–Hilton argument in the context of degenerate higher categories appears as a heirarchy of results:

- Let $\mathcal{A}$ be a set with two monoid structures satisfying interchange, then the Eckmann–Hilton argument shows that the two monoid structures are the same and commutative.
- Let $\mathcal{A}$ be a category with two monoidal structures satisfying interchange, then the *weak* Eckmann–Hilton argument produces a braiding such that the monoidal products are weakly commutative and isomorphic.
- With a third monoidal structure on the category $\mathcal{A}$, the Eckmann–Hilton sphere shows that the braiding is in fact a symmetry.





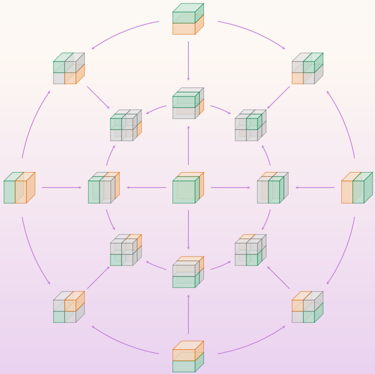
The Periodic Table of n-Categories

set	category	2-category	3-category	4-category
monoid	monoidal category	monoidal 2-category	monoidal 3-category	monoidal 4-category
commutative monoid	braided monoidal category	braided monoidal 2-category	braided monoidal 3-category	braided monoidal 4-category
commutative monoid	symmetric monoidal category	sylleptic monoidal 2-category	sylleptic monoidal 3-category	sylleptic monoidal 4-category
commutative monoid	symmetric monoidal category	symmetric monoidal 2-category	...	...

Summary

We have a heirarchy of results in progress demonstrating that all of the following produce symmetric monoidal categories:

- 3-tuply monoidal categories: two weak, one strict, strict interchanges
- 3-degenerate 4-categories: produced using iconic constructions
- $(n-1)$-degenerate n-categories: produced using iconic constructions



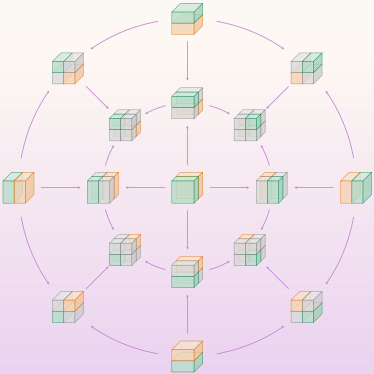
Summary

We have a heirarchy of results in progress demonstrating that all of the following produce symmetric monoidal categories:

- 3-tuply monoidal categories: two weak, one strict, strict interchanges
- 3-degenerate 4-categories: produced using iconic constructions
- $(n-1)$-degenerate n-categories: produced using iconic constructions

Future Work

- Totalities: Do this for triply-degenerate 4-categories and $(n-1)$-degenerate n-categories.
- Combinatorics: Investigate the interesting structures arising from weak interchange and the Eckmann–Hilton sphere.
- Higher Dimensions: Look at the higher Eckmann–Hilton spheres.



Thank you!